Part A

1. $|\psi\rangle = i|1\rangle + 5|2\rangle - 2i|3\rangle$

$\Rightarrow$ rewrite as we did with the lecture 4 example

$$|\psi\rangle = \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix} \xrightarrow[\text{bra}]{\text{find}} \langle\psi| = \begin{pmatrix} -i & 5 & 2i \end{pmatrix}$$

inner product: $\langle\psi|\psi\rangle = \begin{pmatrix} -i & 5 & 2i \end{pmatrix} \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix}$

$$= -i(i) + 5(5) + 2i(-2i)$$

$$= 1 + 25 + 30 \neq 1$$ so not normalized

Normalize by dividing $|\psi\rangle$ by $\sqrt{\langle\psi|\psi\rangle}$ (its magnitude)

$$|\psi\rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix}$$

2) The possible energies are the eigenvalues of the Hamiltonian which are $\hbar\omega/2$, $3\hbar\omega/2$, $5\hbar\omega/2$. $\hbar\omega/2$ corresponds to the eigenvector $|1\rangle$, $3\hbar\omega/2$ to the eigenvector $|2\rangle$, & $5\hbar\omega/2$ to the eigenvector $|3\rangle$. From using an eigenvalue/eigenvector solver we can find that

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If we write the normalized wavefunction in terms of $|1\rangle$, $|2\rangle$, & $|3\rangle$ then we have

$$|\psi\rangle = \frac{i}{\sqrt{30}}|1\rangle + \frac{5}{\sqrt{30}}|2\rangle - \frac{2i}{\sqrt{30}}|3\rangle$$

$$P(\hbar w/2) = \left|\frac{i}{\sqrt{30}}\right|^2 = 1/30 \qquad P(3\hbar w/2) = \left|\frac{5}{\sqrt{30}}\right|^2 = 25/30 \qquad P(5\hbar w/2) = \left|\frac{-2i}{\sqrt{30}}\right|^2 = 4/30$$

So there is a $1/30$ chance to measure the energy & get $\hbar w/2$, a $25/30$ chance to get $3\hbar w/2$, & a $4/30$ chance to get $5\hbar w/2$. Note $P(\hbar w/2) + P(3\hbar w/2) + P(5\hbar w/2) = 1$ as expected

3) $\langle\psi|H|\psi\rangle$

$$= \frac{1}{\sqrt{30}}(-i \;\; 5 \;\; 2i)\,\hbar w \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 5/2 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix}$$

$$= \frac{\hbar w}{30}(-i \;\; 5 \;\; 2i) \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 5/2 \end{pmatrix} \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix}$$

$$= \frac{\hbar w}{30}(-i \;\; 5 \;\; 2i) \begin{pmatrix} i/2 \\ 15/2 \\ -5i \end{pmatrix}$$

$$= \frac{\hbar w}{30}\left[-i\left(\frac{i}{2}\right) + 5\left(\frac{15}{2}\right) + 2i(-5i)\right]$$

$$= \frac{\hbar w}{30}\left(\frac{1}{2} + \frac{75}{2} + 10\right]$$

$$= \frac{\hbar w}{30}(48) = \frac{8}{5}\hbar w = \langle E\rangle$$

4) For $|1\rangle, |2\rangle, |3\rangle$ to be a basis we need to show that $\langle i|j\rangle = \delta_{ij}$ for all possible combinations of i, j

let's start with $i = j$

$\langle 1|1\rangle = (1\ 0\ 0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$ $\quad \langle 3|3\rangle = (0\ 0\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix} = 1$

$\langle 2|2\rangle = (0\ 1\ 0)\begin{pmatrix}0\\1\\0\end{pmatrix} = 1$ $\quad$ so $\langle i|i\rangle = 1$ for all possible i

Now for $i \neq j$

$\langle 1|2\rangle = (1\ 0\ 0)\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$

$\langle 1|3\rangle = (1\ 0\ 0)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

$\langle 2|1\rangle = \langle 1|2\rangle^* = 0$

$\langle 2|3\rangle = (0\ 1\ 0)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

$\langle 3|1\rangle = \langle 1|3\rangle^* = 0$

$\langle 3|2\rangle = \langle 2|3\rangle^* = 0$

so $\langle i|j\rangle = 0$ for all possible i, j so $|1\rangle, |2\rangle, |3\rangle$ have the properties of a basis

5) $\hat{O} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 2\end{bmatrix} \Rightarrow \hat{O}$ is hermitian so it could be an operator

eigenvalue / eigenvector pairs

$$2 \to \begin{pmatrix}0\\0\\1\end{pmatrix} = |\alpha\rangle \qquad -1 \to \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} = |\beta\rangle \qquad 1 \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = |\gamma\rangle$$

$$P(2) = |\langle \alpha | \psi \rangle|^2$$

$$= \left| (0\ 0\ 1) \frac{1}{\sqrt{30}} \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} (-2i) \right|^2 = \frac{4}{30} = P(2)$$

$$P(1) = |\langle \gamma | \psi \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1\ 1\ 0) \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{60}} \left[1(i) + 1(5) \right] \right|^2 = \frac{1}{60} (1 + 25)$$

$$= \frac{26}{60} = \frac{13}{30} = P(1)$$

$$P(-1) = |\langle \beta | \psi \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} (-1\ 1\ 0) \begin{pmatrix} i \\ 5 \\ -2i \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{60}} \left[-1(i) + 1(5) \right] \right|^2 = \frac{1}{60} (1 + 25)$$

$$= \frac{26}{60} = \frac{13}{30} = P(-1)$$

$P(2) + P(1) + P(-1) = 1$ as expected

Part B

1) $|\psi\rangle = \begin{bmatrix} 3i \\ 5 \end{bmatrix} \Rightarrow \langle\psi| = \begin{bmatrix} -3i & 5 \end{bmatrix}$

$\langle\psi|\psi\rangle = \begin{bmatrix} -3i & 5 \end{bmatrix}\begin{bmatrix} 3i \\ 5 \end{bmatrix} = -3i(3i) + 5(5)$
$= 9 + 25 = 34 \neq 1$

$|\psi\rangle = \frac{1}{\sqrt{34}}\begin{bmatrix} 3i \\ 5 \end{bmatrix}$

$|\phi\rangle = \begin{bmatrix} 4i \\ 0 \end{bmatrix} \Rightarrow \langle\phi| = \begin{bmatrix} -4i & 0 \end{bmatrix}$

$\langle\phi|\phi\rangle = \begin{bmatrix} -4i & 0 \end{bmatrix}\begin{bmatrix} 4i \\ 0 \end{bmatrix} = 16 \neq 1$

$|\phi\rangle = \frac{1}{4}\begin{bmatrix} 4i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$

2) $\langle\psi|S_z|\psi\rangle = \frac{1}{\sqrt{34}}\begin{bmatrix} -3i & 5 \end{bmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{34}}\begin{bmatrix} 3i \\ 5 \end{bmatrix}$

$= \frac{\hbar}{68}\begin{bmatrix} -3i & 5 \end{bmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{bmatrix} 3i \\ 5 \end{bmatrix}$

$= \frac{\hbar}{68}\begin{bmatrix} -3i & 5 \end{bmatrix}\begin{bmatrix} 3i \\ -5 \end{bmatrix}$

$= \frac{\hbar}{68}\left(-3i(3i) + 5(-5)\right)$

$= \frac{\hbar}{68}(9 - 25) = \frac{16\hbar}{68} = \frac{8\hbar}{34} = \frac{4\hbar}{17} = \langle\psi|S_z|\psi\rangle$

$\langle\phi|S_z|\phi\rangle = \begin{bmatrix} -i & 0 \end{bmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix}$

$= \frac{\hbar}{2}\begin{bmatrix} -i & 0 \end{bmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} -i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix}$

$= 0 = \langle\phi|S_x|\phi\rangle$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = S_z S_z$$

$$\langle \psi | S_z | \psi \rangle = \frac{4\hbar}{17}$$

$$\langle \phi | S_z | \phi \rangle = [-i \quad 0] \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$= \frac{\hbar}{2}[-i \quad 0]\begin{bmatrix} i \\ 0 \end{bmatrix} = \frac{\hbar}{2} = \langle \phi | S_z | \phi \rangle$$

$$\langle \psi | S_z^2 | \psi \rangle = \frac{1}{\sqrt{34}}[-3i \quad 5] \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{34}}\begin{bmatrix} 3i \\ 5 \end{bmatrix}$$

$$= \frac{\hbar^2}{136}[-3i \quad 5]\begin{bmatrix} 3i \\ 5 \end{bmatrix} = \frac{\hbar^2}{136}(9 + 25)$$

$$= \frac{34\hbar^2}{136} = \frac{\hbar^2}{4} = \langle \psi | S_z^2 | \psi \rangle$$

$$\langle \phi | S_z^2 | \phi \rangle = [-i \quad 0] \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$= \frac{\hbar^2}{4}[-i \quad 0]\begin{bmatrix} i \\ 0 \end{bmatrix} = \frac{\hbar^2}{4} = \langle \phi | S_z^2 | \phi \rangle$$

$$\sigma_{S_z} = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

$$|\psi\rangle: \sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \left(\frac{4\hbar}{17}\right)^2}$$

$$= \frac{15\hbar}{34}$$

$$|\phi\rangle: \sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2} = 0$$

4) the eigenvector of S_y that corresponds to $-\hbar/2$ is $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

ψ: $P(-\hbar/2) = |\langle - | \psi \rangle|^2$

$$= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \end{pmatrix} \frac{1}{\sqrt{34}} \begin{pmatrix} 3i \\ 5 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{68}} \left(-i(3i) + 5 \right) \right|^2$$

$$= \frac{64}{68} = \frac{32}{34} = \frac{16}{17} = P(-\hbar/2, \psi)$$

ϕ: $P(-\hbar/2) = |\langle - | \phi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} \right|^2$

$$= \left| \frac{1}{\sqrt{2}} -i(i) \right|^2 = \frac{1}{2} = P(-\hbar/2, \phi)$$

5) $P(0, \psi) = P(0, \phi) = 0$ since 0 is not an eigenvalue of S_z it is not a possible measured observable