Quantum Mechanics Crash Course (Part 1): Concepts and Statistics

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What Makes Up Matter?

- All matter is made up of atoms and molecules
  - Atoms are made up of electrons and a nucleus
  - The nucleus is made up of protons and neutrons
  - Protons and neutrons are made up of up and down quarks
- Quantum mechanics is the study of things so small we cannot see them, classical physics no longer applies.
- Quantum mechanical systems are not deterministic, they are probabilistic, the results of experiments are statistically random (but still describable!)

Lights and Photons

- In classical physics we think of light as a wave, but in quantum physics we consider it to be made up of particles called photons
- We have proof of this!
    - Photoelectric effect: Light can cause a current in a wire (by freeing electrons) but only certain wavelengths of light
        - Does not depend on intensity

Wave-Particle Duality

- Light has a wave-particle duality $\longrightarrow$ it behaves like a wave but it also behaves like a particle
- This is also true for things so small that quantum mechanics applies to them $\longrightarrow$ they behave like particles but also have properties of a wave.
- We call this wave-particle duality.
  - The consequence of this is the probabilistic nature of quantum mechanics: what is the location of a wave?

Interlude: Probability and Statistics

- Since quantum mechanics is probabilistic, we need to have a quick discussion of probability and statistics
- Discrete vs. Continuous
  - Discrete variables: the data can only have certain values
  - Continuous variables: the data can have any value
- We define the total number as:

$$N = \sum_{j=0}^{\infty} N(j),$$

where N(j) is the number of instances of a certain event.

Discrete Variables Example

- Consider a room containing 14 people of the following ages:
  - one person aged 14
  - one person aged 15
  - three people aged 16
  - two people aged 22
  - two people aged 24
  - five people aged 25
- Then we can define the numbers as:
  - $N(14) = 1$
  - $N(15) = 1$
  - $N(16) = 3$
  - $N(22) = 2$
  - $N(24) = 2$
  - $N(25) = 5$

Probability

- Discrete: the probability of event j occurring is:

$$P(j) = \frac{N(j)}{N}$$

  - Note that $\sum_{j=0}^{\infty} P(j) = 1$
- Continuous: the probability that an event will occur between a and b is:

$$P_{ab} = \int_a^b \rho(x) dx$$

  - $\rho(x)$ is the probability density $\longrightarrow$ the probability of getting x
  - Note that $\int_{-\infty}^{\infty} \rho(x) dx = 1$

Discrete Variables Example

- $N = 14$
- $N(14) = 1$
- $N(15) = 1$
- $N(16) = 3$
- $N(22) = 2$
- $N(24) = 2$
- $N(25) = 5$

- We can define the probabilities as:
  - $P(14) = 1/14 \approx 0.071$
  - $P(15) = 1/14 \approx 0.071$
  - $P(16) = 3/14 \approx 0.214$
  - $P(22) = 2/14 \approx 0.142$
  - $P(24) = 2/14 \approx 0.142$
  - $P(25) = 5/14 \approx 0.357$
- The most probable age is 25 (the highest probability)

Average (Mean) or Expectation Value

- We will denote the average as $\langle \hat{j} \rangle$ (expectation value)
- Discrete:

$$\langle j \rangle = \frac{\sum jN(j)}{N} = \sum_{j=0}^{\infty} jP(j)$$

- Continuous:

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$

Discrete Variables Example

$$\langle j \rangle = 14(\frac{1}{14}) + 15(\frac{1}{14}) + 16(\frac{3}{14}) + 22(\frac{2}{14}) + 24(\frac{2}{14}) + 25(\frac{5}{14})$$

$$\langle j \rangle = 21$$

Other Expectation Values

- Expectation values other than the average: if you replace j/x in the previous slide with a function, you can find the average value of that function over the given data
- Discrete:

$$\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)P(j)$$

- Continuous:

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx$$

Discrete Variables Example

- Let's assume we have some simple formula that relates a person's health insurance costs to their age, a:

$$I = 5a + 7$$

- Then the expectation value for the insurance rate is:

$$\langle I \rangle = (5(14) + 7)(\frac{1}{14}) + (5(15) + 7)(\frac{1}{14}) + (5(16) + 7)(\frac{3}{14})$$

$$+(5(22) + 7)(\frac{2}{14}) + (5(24) + 7)(\frac{2}{14}) + (5(25) + 7)(\frac{5}{14})$$

$$\langle I \rangle = 112.0$$

Standard Deviation

- Standard Deviation
  - We will denote the standard deviation as σ
- Discrete or Continuous:

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Discrete Variables Example

- We previously found that $\langle j \rangle = 21$, so now we have:

$$\langle j^2 \rangle = 14^2(\frac{1}{14}) + 15^2(\frac{1}{14}) + 16^2(\frac{3}{14}) + 22^2(\frac{2}{14}) + 24^2(\frac{2}{14}) + 25^2(\frac{5}{14})$$

$$\langle j^2 \rangle \approx 459.57$$

- Then we can calculate the standard deviation using $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$:

$$\sigma = \sqrt{459.57 - 21^2} = 4.30$$

Events and Likelihoods

- Now that we have the statistics defined, let's work on some new notation
- Let's consider the results of flipping a quarter, Q. There is a 50% chance of getting heads (H) and a 50% chance of getting tails (T).
- We will represent the event of flipping a quarter as $|Q\rangle$ as the sum of its possible outcomes ($|H\rangle$ and $|T\rangle$):

$$|Q\rangle = \sqrt{\frac{1}{2}}|H\rangle + \sqrt{\frac{1}{2}}|T\rangle$$

  - Note that the coefficients have meaning: $|\sqrt{\frac{1}{2}}|^2 = 0.5$, the probability of each event occurring
  - Also note that $|\sqrt{\frac{1}{2}}|^2 + |\sqrt{\frac{1}{2}}|^2 = 1$
    - **The probabilities must add to 1**

Wavefunction

- In the last slide we created a representation for the events that occur when flipping a quarter
- We use this same idea when writing an equation to represent a quantum mechanical state, called a *wavefunction* (typically represented as $|\psi\rangle$)
  - A wavefunction tells you everything that can be known about a quantum mechanical state
  - You extract the information about the state through mathematical operations like *expectation values* and *standard deviations*
- We will go into the math more on Monday, but the ket notation should give you a hint of the form

Spin

- All particles have certain properties which define what type of particle it is: mass and charge
- Particles of the same type of a property that can change without changing the particle type: spin
  - *Degree of Freedom*
  - Spin can be ± 1 and be in the x, y, and z direction
  - A qubit can be defined with the z-spin: up $(+1)$ or down (-1)

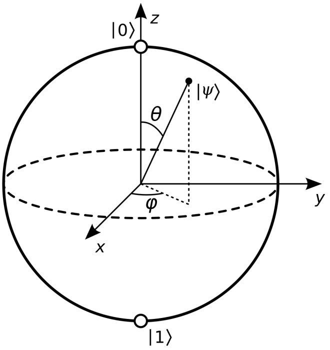

Figure 1: Bloch Sphere

Spin Up and Spin Down

- There are several ways to represent spin-up and spin-down quantum states but we will start this class using an intuitive one:
  - Spin-up: $|\uparrow\rangle$
  - Spin-down: $|\downarrow\rangle$
- Let's consider the following *quantum spin state*

$$|\psi\rangle = \frac{4}{5}|\uparrow\rangle - \frac{3i}{5}|\downarrow\rangle$$

Superposition

$$|\psi\rangle = \sqrt{\frac{4}{5}}|\uparrow\rangle - \sqrt{\frac{3i}{5}}|\downarrow\rangle$$

- The above state is in a superposition of two states: $|\uparrow\rangle$ and $|\downarrow\rangle$
  - We can think of it as being simulatineously spin-up and spin-down.
- Most quantum states we encounter in this class will be a superposition of two or more states
  - Remember that the coefficients are related to the probabilities of each state

Measurement

- Any interaction strong enough to measure a quantum mechanical system is strong enough to change it
- The first measurement of a system *prepares* the system for additional measurements

$$|\psi\rangle = \frac{4}{5}|\uparrow\rangle - \frac{3i}{5}|\downarrow\rangle$$

* When the above spin state is measured, it will *collapse* into either $|\uparrow\rangle$ or $|\downarrow\rangle$ (it will no longer be in a superposition)

Operators and Observables

- When we measure a quantum state, we say that we are applying an *operator* to the state and obtaining an *observable*.
  - For example, the momentum operator is applied to a quatum state and the returned observable is the momentum of the state.
- Only certain states and observables are possible for a given operator, we will discuss why on Wednesday.

Example: Superposition, Measurement, and State Preparation

An operator $\hat{A}$, representing observable A, has two normalized states ψ_1 and ψ_2, with values a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two normalized states ψ_1 and ψ_2 with values b_1 and b_2. The states are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5 \quad \psi_2 = (4\phi_1 - 3\phi_2)/5$$

a. Observable A is measured, and the value a_1 is obtained. What is the state of the system immediately after this measurement?
b. If B is now measured, what are the possible results, and what are their probabilities?

Entanglement

- In the previous example, the states of $\hat{A}$ and the states of $\hat{B}$ are *entangled*, what happens to one state will change or affect what happens to the other state.
  - Measuring (and thus collapsing) the states of $\hat{A}$ change the probabilities of the measurements of $\hat{B}$

Uncertainty

- Certain pairs of observables can be measured together, but only to a set *uncertainity*
  - Example: position and momentum can be measured together, but only to within $\hbar/2$
    - $\sigma_x \sigma_p \geq \hbar/2$
    - Increasing the precision of the position measurement decreases the position of the momentum measurement
- In quantum computing, when we want to measure various attributes of a quantum system, we need to be careful that we are not measuring observables simultaneously if they have an *uncertainity principle*

Energy

- In quantum mechanics, energy is a *quantized* observable: it can only have certain values
  - Spin is another quantized observable

References and Resources

- *Quantum Mechanics: The Theoretical Minimum*. Leonard Susskind.
- *Introduction to Quantum Mechanics*. David J. Griffiths and Darrell F. Schroeter. 3rd. ed.
- If You Don't Understand Quantum Physics, Try This! (Video)