More Quantum Fourier Transforms, Quantum Phase Estimation, and Quantum Parallelism

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Imaginary and Complex Numbers (Part 2)

Representing a Complex Number as a Vector

To the board!

The Three Ways to Represent a Complex Number

- Normal: a+ib
- Polar: r(cosθ + isinθ)
  - $r = |a + ib| = \sqrt{a^2 + b^2}$
  - $\theta = arctan\frac{b}{a}$
- Exponential: $re^{i\theta}$
- Remember Euler's Identity: $e^{i\theta} = cos\theta + isin\theta$

Examples

Convert the followng complex numbers to polar and exponential form. Draw the complex number on the complex plane.

- 3+2i
- -4+i
- 5-2i
- -9-3i

The Math of the Quantum Fourier Transform (Sorry, we have to do it to learn :( )

Quantum Circuit to Represent 5

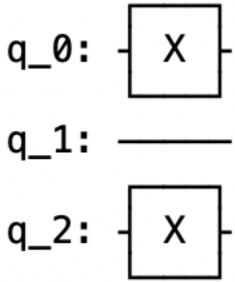


Figure 1: Quantum Circuit to Represent 5

Binary 5 with Bloch Spheres

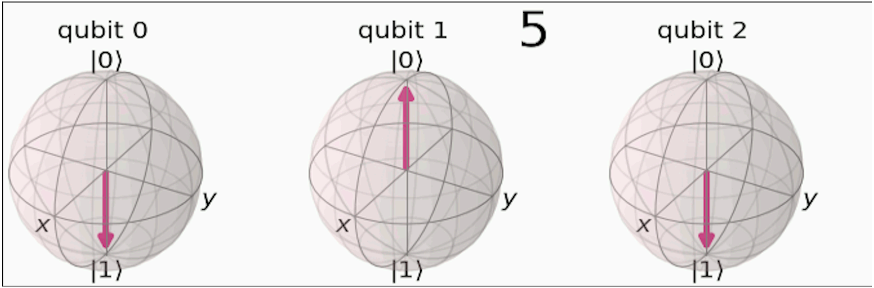


Figure 2: Binary 5 with Bloch Spheres

Quantum Fourier Transform on 5: Circuit

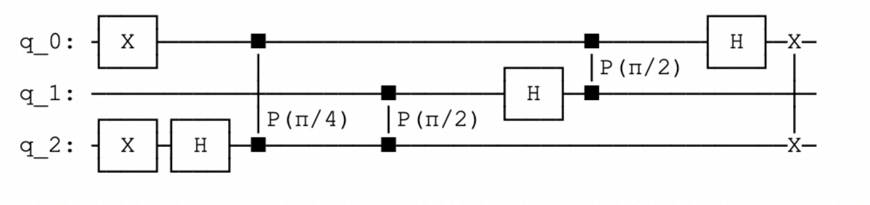


Figure 3: Quantum Fourier Transform on 5: Circuit

Quantum Fourier Transform on 5: Bloch Spheres

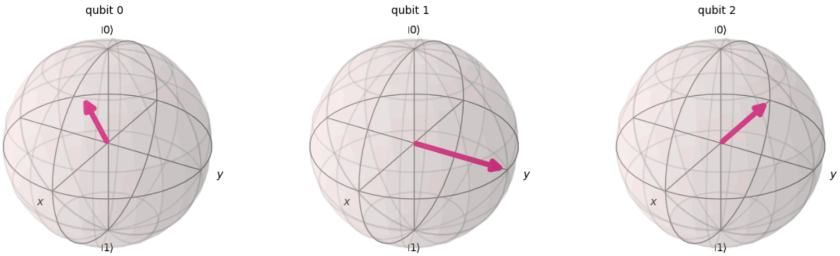


Figure 4: Quantum Fourier Transform on 5: Bloch Spheres

But can we determine how much each qubit has rotated?

- Remember that the Hadamard gates but the qubit at the center of the Bloch sphere
  - QFT rotates the qubit around the xy-plane by adding a phase
- Leftmost qubit (qubit 0) is rotated by $\frac{5}{2^n} = \frac{5}{2^3} = \frac{5}{8}$ full turns ($\frac{5}{8} * 2\pi = \frac{5\pi}{4}$ radians).
- Middle qubit (qubit 1) is rotated by double this ($\frac{5\pi}{4} * 2 = \frac{5\pi}{2}$ radians).
- Rightmost qubit (qubit 2) is double the rotation of qubit 1 ($\frac{5\pi}{2} * 2 = 5\pi$ radians)
- But what does this mean?

Let's Determine the Phases

- Remember that the phase gate applies a phase angle of $\phi = \frac{\pi}{2^k}$, where k is the relative distance between the qubits
- Remember the phase gate is:

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

- Now let us write our binary 5 state as: $|5\rangle = |101\rangle = |k_0 k_1 k_2\rangle$

A New Way to Think About Hadamard Gates

- Remember that:

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Now consider the following representation, where k is either 0 or 1:

$$H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k|1\rangle)$$

- This is a general way to represent the Hadamard gate if the exact state is not known

Slightly More Manipulation

$$H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k|1\rangle)$$

is equivalent to

$$H|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi k}|1\rangle)$$

Since $e^{i\pi k} = cos(\pi k) + isin(\pi k)$ which is 1 if k = 0 and -1 if k = 1

Back to The Circuit

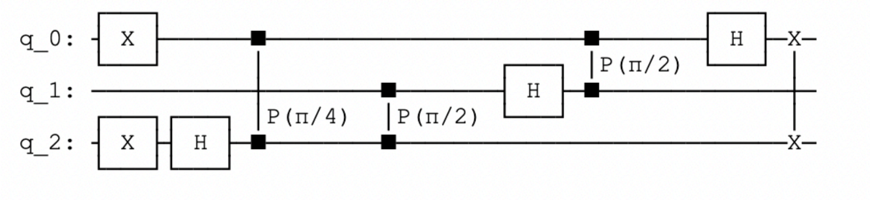


Figure 5: Quantum Fourier Transform on 5: Circuit

- Qubit 2 goes through a Hadamard gate and then two controlled phase gates
  - Remember that phase gates only apply to $|1\rangle$

Determining Phases of Qubit 2

- Thus for qubit 2 (which is a $|1\rangle$ and will be qubit 0 after the SWAP), we have:

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi}|1\rangle)$$

$$P(\frac{\pi}{4})H|1\rangle = \frac{1}{\sqrt{2}}(P(\frac{\pi}{4})|0\rangle + e^{i\pi}P(\frac{\pi}{4})|1\rangle)$$

$$P(\frac{\pi}{4})H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi}e^{i\pi/4}|1\rangle$$

Determining Phases of Qubit 2 (Continued)

$$P(\frac{\pi}{4})H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{5\pi}{4}}|1\rangle)$$

- Not that the $P(\frac{\pi}{2})$ gate does not trigger since $|k_1\rangle = |0\rangle$
- Does $\frac{5\pi}{4}$ sound familiar??

Moving Onto Qubit 1

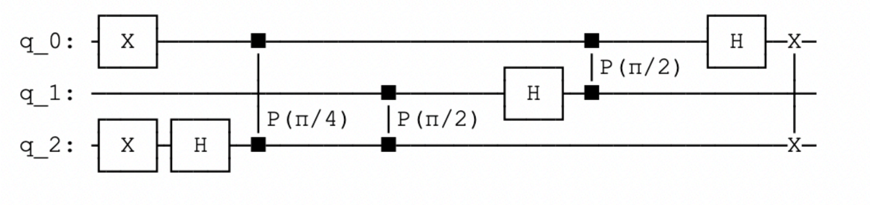


Figure 6: Quantum Fourier Transform on 5: Circuit

- Qubit 1 goes through a Hadamard gate and then one controlled phase gates
  - Remember that phase gates only apply to $|1\rangle$

Phase of Qubit 1

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{0}|1\rangle)$$

$$P(\frac{\pi}{2})H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{2}}|1\rangle)$$

- $\frac{\pi}{2}$ seem familar (or an equivalent angle)?

Phase of Qubit 0

- Qubit 0 (which is the last qubit after the SWAP) is $|1\rangle$ and only has a Hadamard gate

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi}|1\rangle)$$

- π seem familar (or an equivalent angle)?
- Note: You can take the three states we just derived and perform tensor products on them to create the three qubit state. The results (with some amount of algebra) will be the same as printed in Qiskit with the Statevector function.

Important Note: We calculated the phases using the state of each qubit $\longrightarrow$ Infomation about the states of all qubits are encoded into the phases during QFT

Qiskit built-in QFT Function

```
from qiskit.circuit.library import QFT

# Create a 3-qubit Quantum Circuit
qc = QuantumCircuit(3)

# Apply a QFT on the qubits
qft_circuit = QFT(num_qubits=3)
qc.append(qft_circuit.to_instruction(), range(3))
```

And Inverse QFT Function

- **Reminder: All Quantum Circuits Are Reversible**

```
# Apply a QFT on the qubits
qft_circuit = QFT(num_qubits=3, inverse=True)
qc.append(qft_circuit.to_instruction(), range(3))

# Draw the circuit
qc.draw(output='mpl')
```

OFT Conclusion

- QFT or Inverse QFT will come up very often, use either your circuit to see all the gates (you can build one for IQFT) or use the built-in Qiskit function for the "black-box"
- Is it better than DFT? For an N digit binary number:
  - The number of gates on QFT grows as $O(N^2)$
  - The number of operations on DFT grows as $O(N2^N)$
- Not quite the whole picture...QFT becomes better at 22 qubits/bits or at more than 4 million data points to process
  - Big Data?

Quantum Parallelism

What is Parallel Computing in a Classical Sense?

- Using many processors (either across multiple computers or on the same computer) to solve the same task.
- Common paradigm in computing that is used to speed up calculations by dividing the work among different processors
- Works very well for some problems, not as well for others, and does run into communication overheads at large scales

Quatum Parallelism

- Quantum parallelism is not about using multiple quantum computers to solve a single problem
- Instead, it refers to a quantum circuit being in a superposition of all possible results at the same time
  - Explores the search space
- Consider three qubits with Hadamard gates on each qubit. Possible results?
  - If I apply gates to this circuit, I am applying the gates to all possible states the circuit could be in **at the same time**
- Note that it may not lead to speed-up

Parallel Quantum Computers?

- Perhaps in the future, but definitely not right now.

Quantum Phase Estimation

Quantum Phase Estimation (QPE)

- Key part of many other larger quantum algorithms, including Shor's algorithm

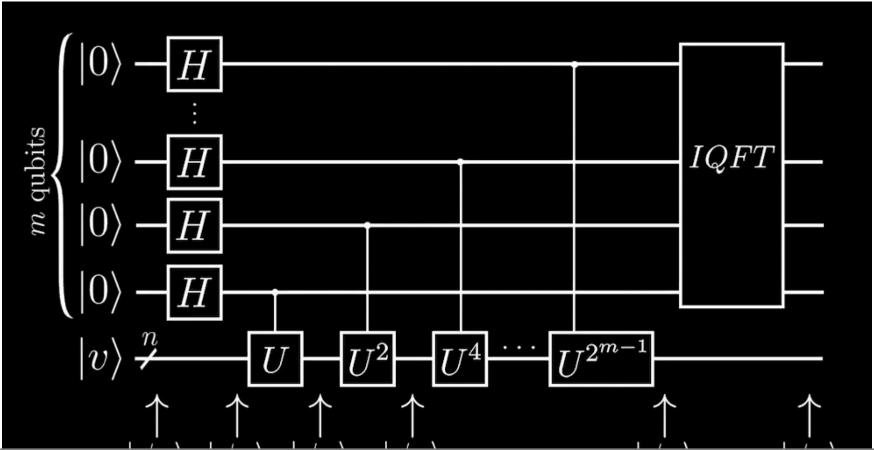
Problem Statement

- The quantum phase estimation algorithms solves the following problem by finding θ:

$$U|v\rangle = e^{i\theta}|v\rangle$$

- U is a matrix
- $|v\rangle$ is an eigenvector of the matrix
- $e^{i\theta}$ is the corresponding eigenvalue

Circuit Setup



New Gate and Notation: U

- Here U is any unitary 2x2 matrix (would not be that fun if it had a set form...)
- U is controlled by a control qubit, normal rules apply
- U in the circuit diagram means U is applied once
- U^2 in the circuit diagram means U is applied twice
- U^4 means U is applied four times, and so on

Phase Kickback Review

- What if the target and control qubits are superpositions?

$$C_X|-+\rangle = \frac{1}{2}(C_X|\uparrow\uparrow\rangle + C_X|\uparrow\downarrow\rangle - C_X|\downarrow\uparrow\rangle - C_X|\downarrow\downarrow\rangle)$$

$$= \frac{1}{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

$$= \frac{1}{2}(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

$$|--\rangle$$

- ...we changed the control qubit, not the target qubit...

Phase Kickback: Updated

- Let's think about this another way. The NOT gate has the following matrix representation with the following eigenvectors

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- The first eigenvector has an eigenvalue of -1, the second has an eigenvalue of 1
- Do the eigenvectors look familar? $|+\rangle$ and $|-\rangle$!
- If the target qubit is $|-\rangle$ (an eigenvector) then the CNOT gate has a phase kickback that adds a factor of -1 (the eigenvalue) to the control qubit

Phase Kickback: Updated

- If a controlled gate is a applied to a system such that the target qubit is an eigenvector of the gate, then the target qubit will not be changed. Instead, the eigenvalue corresponding to the eigenvector will be added as a phase to the control qubit.

Phase Kickback with the U Gate

- Since $|v\rangle$ is an eigenvector of U, each of the controlled U gates in the circuit adds a factor of $e^{i\theta}$ to the control qubit (U^2 adds two factors, U^4 adds four factors, ...)

Walking Through the Circuit

- At location 0 the circuit would be:

$$|\psi_0\rangle = |0\rangle^{\otimes m}|v\rangle = |0\rangle \otimes |0\rangle \otimes ... \otimes |0\rangle|v\rangle$$

- Remember that $\otimes$ is the tensor product (entanglement)
- Here $|v\rangle$ can be $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, or any other state as long as it is an eigenvector of U.

Walking Through the Circuit (Continued)

- $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so at location 1:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes ... \otimes (|0\rangle + |1\rangle))|v\rangle$$

Walking Through the Circuit (Continued Continued)

- Location 2: U gate is applied with the target being the control qubit, but the control being the highest indexed qubit
- Since $|v\rangle$ is a eigenvector of U, phase kickback applies a factor of $e^{i\theta}$ to the control qubit ($|1\rangle$) state only

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes ... \otimes (|0\rangle + e^{i\theta}|1\rangle))|v\rangle$$

Walking Through the Circuit (Continued X3)

- Location 3 has the U gate applied twice where the target is $|v\rangle$ and the control is the second highest indexed qubit, so there is a double phase kickback

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes ...$$

$$\otimes(|0\rangle + e^{i\theta}e^{i\theta}|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle))|v\rangle$$

$$= \frac{1}{\sqrt{2}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes ...$$

$$\otimes(|0\rangle + e^{2i\theta}1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle))|v\rangle$$

Walking Through the Circuit (Continued X4)

- This process continues where each lower indexed qubit will pick up one extra factor of $e^{i\theta}$ so at location 4 we have

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2^{m-1}i\theta}|1\rangle) \otimes (|0\rangle + e^{2^{m-2}i\theta}|1\rangle) \otimes ...$$

$$\otimes(|0\rangle + e^{2i\theta}|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle))|v\rangle$$

Further Substitutions and Manipulations

- Now we need to make some assumptions about θ
- Let's assume that $\theta = 2\pi j$ for some value of j, but j is a binary number
  - $j = 0.j_0 j_1 j_2 ... j_{m-1}$ which each j_i is either 0 or 1 and the leading "0." indicates that this number is less than or equal to 1
  - Any angle can be represented as 2π times some number less than or equal to 1

Substitute in the Binary Digit

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2\pi 2^{m-1} i(j)}|1\rangle) \otimes (|0\rangle + e^{2\pi 2^{m-2} i(j)}|1\rangle) \otimes ...$$

$$\otimes(|0\rangle + e^{2\pi 2 i(j)}|1\rangle) \otimes (|0\rangle + e^{2\pi i(j)}|1\rangle))|v\rangle$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2\pi 2^{m-1} i(j_0 j_1 j_2 \cdots j_{m-1})}|1\rangle)\otimes$$

$$(|0\rangle + e^{2\pi 2^{m-2} i(j_0 j_1 j_2 \cdots j_{m-1})}|1\rangle) \otimes ...$$

$$\otimes(|0\rangle + e^{2\pi 2 i(j_0 j_1 j_2 \cdots j_{m-1})}|1\rangle) \otimes (|0\rangle + e^{2\pi i(j_0 j_1 j_2 \cdots j_{m-1})}|1\rangle))|v\rangle$$

Back to the Binary String

- $j = 0.j_0 j_1 j_2 ... j_{m-1}$ which each j_i is either 0 or 1 and the leading "0." indicates that this number is less than or equal to 1
- j can be converted to Base-10 using the following equation:

$$j = \frac{j_0}{2} + \frac{j_1}{4} + \frac{j_2}{8} + ... + \frac{j_m - 1}{2^m}$$

Substituting into the State Gives...

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2\pi 2^{m-1} i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+\ldots+\frac{j_{m-1}}{2^m})}|1\rangle)\otimes$$

$$(|0\rangle + e^{2\pi 2^{m-2} i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+\ldots+\frac{j_{m-1}}{2^m})}|1\rangle) \otimes \ldots$$

$$\otimes(|0\rangle+e^{2\pi 2 i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+\ldots+\frac{j_{m-1}}{2^m})}|1\rangle)\otimes(|0\rangle+e^{2\pi i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+\ldots+\frac{j_{m-1}}{2^m})}|1\rangle))|v\rangle$$

Distribute the Extra Factors of 2...

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2\pi i(2^{m-2}j_0+2^{m-3}j_1+2^{m-4}j_2+\ldots+\frac{j_{m-1}}{2})}|1\rangle)\otimes$$

$$(|0\rangle + e^{2\pi i(2^{m-3}j_0+2^{m-4}j_1+2^{m-5}j_2+\ldots+\frac{j_{m-1}}{4})}|1\rangle) \otimes \ldots$$

$$\otimes(|0\rangle+e^{2\pi i(j_0+\frac{j_1}{2}+\frac{j_2}{4}+\ldots+\frac{j_{m-1}}{2^{m-1}})}|1\rangle)\otimes(|0\rangle+e^{2\pi i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+\ldots+\frac{j_{m-1}}{2^m})}|1\rangle))|v\rangle$$

Now to Start Reducing Things...

- $2^n j$ where j is either 0 or 1 will give 0 (so not contribute to the sum) or an integer multiple of 2π (not a useful angle)
- Keep only the terms which are fractions:

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}((|0\rangle + e^{2\pi i(\frac{j_{m-1}}{2})}|1\rangle)\otimes$$

$$(|0\rangle + e^{2\pi i(\frac{j_{m-2}}{2}+\frac{j_{m-1}}{4})}|1\rangle) \otimes ...$$

$$\otimes(|0\rangle+e^{2\pi i(\frac{j_1}{2}+\frac{j_2}{4}+...+\frac{j_{m-1}}{2^{m-1}})}|1\rangle)\otimes(|0\rangle+e^{2\pi i(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}+...+\frac{j_{m-1}}{2^m})}|1\rangle))|v\rangle$$

Look familar?

- Each rotation is double the previous rotation...
  - This is the same as if we did the QFT on state j

We can get j!

$$|\psi_5\rangle = IQFT|\psi_4\rangle = |j\rangle$$

- But $\theta = 2\pi j$ and $e^{i\theta}$ is an eigenvalue of U, so by determining $|j\rangle$ we have found the eigenvalue of m (at least to m bits)

Qiskit Time!!