Example 1

$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle$

$|\beta\rangle = i|1\rangle + 2|3\rangle$

$= i|1\rangle + 0 + 2|3\rangle$ ⇒ add a spot for the $|2\rangle$ state with a coefficient of zero

Let's convert $|\alpha\rangle$ & $|\beta\rangle$ into vectors where the first element is the coefficient of $|1\rangle$, the second element is the coefficient of $|2\rangle$, & the last element is the coefficient of $|3\rangle$

$|\alpha\rangle = \begin{pmatrix} i \\ -2 \\ -i \end{pmatrix}$ $\quad |\beta\rangle = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$

Now to find $\langle\alpha|$ & $\langle\beta|$ we take the complex conjugate transpose of $|\alpha\rangle$ & $|\beta\rangle$

$\langle\alpha| = (|\alpha\rangle^*)^T = (-i \;\; -2 \;\; i)$

$\langle\beta| = (|\beta\rangle^*)^T = (-i \;\; 0 \;\; 2)$

Now we need to check to see if each state is normalized by square rooting the inner product of each state with itself

magnitude $|\alpha\rangle$: $\sqrt{\langle\alpha|\alpha\rangle} = \sqrt{(-i \;\; -2 \;\; i)\begin{pmatrix} i \\ -2 \\ -i \end{pmatrix}}$

inner product is dot product

$= \sqrt{-i(i) - 2(-2) + i(-i)}$

$= \sqrt{-(-1) + 4 - (-1)} = \sqrt{6}$ ⇒ not 1 so $|\alpha\rangle$ is not normalized

Normalize $|\alpha\rangle$ by dividing every element of $|\alpha\rangle$ & $\langle\alpha|$ by its magnitude

$$|\alpha\rangle = \begin{pmatrix} i/\sqrt{6} \\ -2/\sqrt{6} \\ -i/\sqrt{6} \end{pmatrix} \qquad \langle\alpha| = \begin{pmatrix} \frac{-i}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{i}{\sqrt{6}} \end{pmatrix}$$

Now let's see if $|\beta\rangle$ is normalized & if it is not then we will normalize it

magnitude of $|\beta\rangle$: $\sqrt{\langle\beta|\beta\rangle} = \sqrt{\begin{pmatrix} -i & 0 & 2 \end{pmatrix}\begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}}$

$$= \sqrt{i(-i) + 0(0) + 2(2)}$$

$$= \sqrt{-(-1) + 4} = \sqrt{5}$$

not 1 so not normalized

$$\Rightarrow |\beta\rangle = \begin{pmatrix} i/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix} \qquad \langle\beta| = \begin{pmatrix} \frac{-i}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Remember that we always want to ensure that our wavefunctions are normalized after we find the ket and bra because the coefficients are related to the probability of the wavefunction collapsing into each of the basis states

Now that we have $\langle\beta|$ and $\langle\alpha|$ as bra vectors, we want to convert them back into a sum of the basis stats as the kets were given. We will do this by making the first element the coefficient if the $|1\rangle$ state, the second element the coefficient of the $|2\rangle$ state, and the final element the coefficient of the $|3\rangle$ state.

$$\langle\alpha| = \begin{pmatrix} \frac{-i}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{i}{\sqrt{6}} \end{pmatrix} = \frac{-i}{\sqrt{6}}\langle 1| - \frac{2}{\sqrt{6}}\langle 2| + \frac{i}{\sqrt{6}}\langle 3|$$

Note that the basis states must be vectors too

$\langle\beta| = \left(\frac{i}{\sqrt{5}} \quad 0 \quad \frac{2}{\sqrt{5}}\right) = \frac{i}{\sqrt{5}}|1\rangle + 0|2\rangle + \frac{2}{\sqrt{5}}|3\rangle$

$= \frac{i}{\sqrt{5}}|1\rangle + \frac{2}{\sqrt{5}}|3\rangle$

Now to compute $\langle\alpha|\beta\rangle$ & $\langle\beta|\alpha\rangle$. Here we will want the states in their vector form

$\langle\alpha|\beta\rangle = \left(\frac{-i}{\sqrt{6}} \quad \frac{-2}{\sqrt{6}} \quad \frac{i}{\sqrt{6}}\right)\begin{pmatrix} i/\sqrt{5} \\ 0 \\ 2/\sqrt{5}\end{pmatrix}$

make sure to use normalized states, now do the dot product

$= \frac{-i}{\sqrt{6}}\left(\frac{i}{\sqrt{5}}\right) - \frac{2}{\sqrt{6}}(0) + \frac{i}{\sqrt{6}}\left(\frac{2}{\sqrt{5}}\right)$

$= \frac{1}{\sqrt{30}} + \frac{2i}{\sqrt{30}} = \frac{1+2i}{\sqrt{30}} = \langle\alpha|\beta\rangle$

$\langle\beta|\alpha\rangle = \left(\frac{-i}{\sqrt{5}} \quad 0 \quad \frac{2}{\sqrt{5}}\right)\begin{pmatrix} i/\sqrt{6} \\ -2/\sqrt{6} \\ -i/\sqrt{6}\end{pmatrix}$

do the dot product

$= \frac{-i}{\sqrt{5}}\left(\frac{i}{\sqrt{6}}\right) + 0\left(\frac{-2}{\sqrt{6}}\right) + \frac{2}{\sqrt{5}}\left(\frac{-i}{\sqrt{6}}\right)$

$= \frac{1}{\sqrt{30}} + \frac{-2i}{\sqrt{30}} = \frac{1-2i}{\sqrt{30}} = \langle\beta|\alpha\rangle$

Futhermore from the above results note that

$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$ $\Rightarrow$ this is a general finding for all inner products!