Example 2

$H = \hbar w \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $A = \lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$B = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

The possible energies of the system are the eigenvalues of the Hamiltonian, so using an eigenvalue solver we get

$E_1 = \hbar w, \quad E_2 = 2\hbar w, \quad E_3 = 2\hbar w$

To find a we need to find $\langle\psi|$ and then take the inner product

$|\psi\rangle = a \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix} \Rightarrow \langle\psi| = a^*[i \;\; 3 \;\; -2i]$

↳ complex conjugate for bra

$\langle\psi|\psi\rangle = 1$ we want to find a so this is true

$a^*[i \;\; 3 \;\; -2i]\, a \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix} = 1$

$a^* a\, [i \;\; 3 \;\; -2i] \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix} = 1$

↳ gather a terms

also $a^* a = |a|^2$

$|a|^2 [i \;\; 3 \;\; -2i] \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix} = 1$

$|a|^2 (i(-i) + 3(3) + -2i(2i)) = 1$

$|a|^2 (1 + 9 + 4) = 1$

$|a|^2 (14) = 1$

$|a|^2 = 1/14$

$a = \sqrt{\frac{1}{14}}$

So our normalized wavefunction is

$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix}$$

with the corresponding bra being

$$\langle\psi| = 1/\sqrt{14} \begin{bmatrix} i & 3 & -2i \end{bmatrix}$$

Now let's find $\langle\psi|A|\psi\rangle$

$$\frac{1}{\sqrt{14}} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{\sqrt{14}} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix}$$

first gather all the scalars

$$= \frac{\lambda}{14} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix}$$

do this matrix vector multiplication first with Wolfram alpha

$$= \frac{\lambda}{14} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \begin{bmatrix} 3 \\ -i \\ 4i \end{bmatrix}$$

Now take the dot product of the two vectors

$$= \frac{\lambda}{14} \left(\cancel{i(3)} + \cancel{3(-i)} - 2i(4i)\right)$$

$$= \frac{\lambda}{14}(8) = \frac{8\lambda}{14} = \boxed{\frac{4\lambda}{7} = \langle\psi|A|\psi\rangle}$$

Now for $\langle\psi|B|\psi\rangle$

$$\frac{1}{\sqrt{14}} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{14}} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix}$$

Combine scalars

$$= \frac{u}{14} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix}$$ multiply matrix & second vector

$$= \frac{u}{14} \begin{bmatrix} i & 3 & -2i \end{bmatrix} \begin{bmatrix} -2i \\ 2i \\ 3 \end{bmatrix}$$ dot product with the two vectors

$$= \frac{u}{14} \left(i(-2i) + \cancel{3(2i)} - \cancel{2i(3)} \right)$$

$$= \frac{u}{14}(2) = \frac{2u}{14} = \boxed{\frac{u}{7} = \langle \psi | B | \psi \rangle}$$

Both $\langle \psi | A | \psi \rangle$ & $\langle \psi | B | \psi \rangle$ are expectation values meaning they are the average result of many observations but not necessarily a possible value for a measurement

Finally the eigenvectors of H are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, & $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

with the first corresponding to the eigenvalue/energy $\hbar\omega$ & the second two corresponding to the eigenvalue/energy $2\hbar\omega$

↳ use wolfram or python to figure this out

↱ general rule

The eigenvectors of our Hamiltonian form our computational basis so let's rewrite $|\psi\rangle$ using these eigenvectors

$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} -i \\ 3 \\ 2i \end{bmatrix} = \frac{-i}{\sqrt{14}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{\sqrt{14}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{2i}{\sqrt{14}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

↳ this is the eigenvector that corresponds to $\hbar\omega$

Since the coefficient in front of the eigenvector that corresponds to the $\hbar\omega$ energy state is $c = -i/\sqrt{14}$, then the probability of measuring the energy of the system & getting $\hbar\omega$ is $P(\hbar\omega) = |c|^2 = \left|\frac{-i}{\sqrt{14}}\right|^2 = \frac{-i}{\sqrt{14}}\left(\frac{i}{\sqrt{14}}\right) = \frac{1}{14} = P(\hbar\omega)$