Example 3

$$|\psi\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\2\end{bmatrix}$$

$\Rightarrow$ check to see if it is normalized

$$\langle\psi| = \frac{1}{\sqrt{6}}\begin{bmatrix}1-i & 2\end{bmatrix}$$

$$\langle\psi|\psi\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}1-i & 2\end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\2\end{bmatrix}$$ gather scalars

$$= \frac{1}{6}\begin{bmatrix}1-i & 2\end{bmatrix}\begin{bmatrix}1+i\\2\end{bmatrix}$$ dot product

$$= \frac{1}{6}\left((1-i)(1+i) + 2(2)\right)$$ FOIL

$$= \frac{1}{6}\left(1 - i + i - (i^2) + 4\right)$$ simplify

$$= \frac{1}{6}(1+1+4) = \frac{6}{6} = 1$$ $\Rightarrow$ normalized since inner product is 1

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$ $\Rightarrow$ found in the slides

let's start by finding the eigenvalue & eigenvector pairs for both matrices

S_x: $\frac{\hbar}{2}, \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$; $-\frac{\hbar}{2}\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

S_z: $\frac{\hbar}{2}, \begin{pmatrix}1\\0\end{pmatrix}$; $-\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$

find with wolfram alpha or Python $\Rightarrow$ make sure the eigenvectors are normalized

Now we need to use the fact that if we measure a specific eigenvalue of an operator then our wavefunction has collapsed into the state represented by the eigenvector

S_x: let $|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ & $|-\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

where $|+\rangle$ corresponds to the eigenvalue $\hbar/2$ & $|-\rangle$ corresponds to the eigenvalue $-\hbar/2$

Therefore the probability of measuring S_x & getting $\hbar/2$ is

$$P(\hbar/2) = |\langle + | \psi \rangle|^2 = \left| \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} \left(\frac{1}{\sqrt{2}}(1+i) + \frac{1}{\sqrt{2}}(2) \right) \right|^2 = \left| \frac{1}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{2}{\sqrt{2}} \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} \left(\frac{3}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \right|^2 = \left| \frac{3}{\sqrt{12}} + \frac{i}{\sqrt{12}} \right|^2 = \frac{9}{12} + \frac{1}{12} = \frac{10}{12} = \boxed{\frac{5}{6} = P(\hbar/2)}$$

↳ for $z = a + ib$, $|z|^2 = a^2 + b^2$

Then the probability of measuring S_x & getting $-\hbar/2$ is

$$P(-\hbar/2) = |\langle - | \psi \rangle|^2 = \left| \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} \left(\frac{-1}{\sqrt{2}}(1+i) + \frac{2}{\sqrt{2}} \right) \right|^2 = \left| \frac{1}{\sqrt{6}} \left(\frac{-1}{\sqrt{2}} \frac{-i}{\sqrt{2}} + \frac{2}{\sqrt{2}} \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} \frac{-i}{\sqrt{2}} \right) \right|^2 = \left| \frac{1}{\sqrt{12}} \frac{-i}{\sqrt{12}} \right|^2 = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \boxed{\frac{1}{6} = P(-\hbar/2)}$$

Note that $P(\hbar/2) + P(-\hbar/2) = 1$ which is required since $\hbar/2$ & $-\hbar/2$ are the only possible outcomes of a measurement $\Rightarrow$ they are the only eigenvalues of S_x.

Do the same process for S_z where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P(\hbar/2) = |\langle 0|\psi\rangle|^2 = \left| (1 \;\; 0) \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{6}} \left(1(1+i) \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} (1+i) \right|^2 = \left| \frac{1}{\sqrt{6}} + \frac{i}{\sqrt{6}} \right|^2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} = P(\hbar/2)$$

$$P(-\hbar/2) = |\langle 1|\psi\rangle|^2 = \left| (0 \;\; 1) \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{6}} (1)(2) \right|^2$$

$$= \left| \frac{2}{\sqrt{6}} \right|^2 = \frac{4}{6} = \frac{2}{3} = P(-\hbar/2)$$

Note again that $P(\hbar/2) + P(-\hbar/2) = 1$ as expected